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The Solution of the d -Dimensional Twisted Group Lattices

Olaf Lechtenfeld* and Stuart Samuel†

**Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
E-mail: lechtenf@itp.uni-hannover.de*

*† Department of Physics
City College of New York
New York, NY 10031 USA
E-mail: samuel@scisun.sci.ccny.cuny.edu*

Abstract

The general d -dimensional twisted group lattice is solved. The irreducible representations of the corresponding group are constructed by an explicit procedure. It is proven that they are complete. All matrix representation solutions to the quantum hyperplane equations are obtained.

I. Introduction

The general d -dimensional twisted group lattice is a particular kind of group lattice. [1, 2] A group lattice is a lattice constructed from a discrete group G and a subset NN of G . The subset NN is called the nearest-neighbor set and is not necessarily a subgroup. Recall that a lattice is determined by specifying sites and bonds. For each element $g \in G$, one associates a site. Hence, the number of lattice sites is the number of elements of G . This is the order of G and is denoted by $o(G)$. The identity element e of G is by definition associated with the origin of the lattice. The subset NN is used to determine the bonds of the lattice. A site g' is a nearest-neighbor site of g if $g'g^{-1} \in NN$. Hence, the nearest-neighbor sites of a site g are the elements in the right-coset NNg , so that the bonds \mathcal{B} of the lattice are $\mathcal{B} = \{[hg, g], \text{ such that } g \in G, h \in NN\}$. Here, the pair $[hg, g]$ indicates that a bond goes between sites hg and g . One requires that if $h \in NN$ then $h^{-1} \in NN$ so that if g' is a nearest neighbor of g then g is also a nearest neighbor of g' . For more discussion on group lattices, see Refs. [1]–[4].

Group lattices are interesting for the following reason. If the irreducible representations of G are known, then one can compute the free propagation of a particle through the lattice. The particle is allowed to move between nearest-neighbor sites. Hopping parameters are introduced to control the ease or difficulty of moving over a bond. The formalism is given in Ref. [1] both for bosons or fermions. It is also possible to define field theories on group lattices. The free field theory is exactly solvable and a perturbative expansion exists for interacting theories. For details of the solution method, see Ref. [1].

The group lattice is a general concept and is expected to have many physical applications. An example occurs for C_{60} . The carbon atoms sit at the sites of a group lattice based on the alternating group A_5 . Using some approximations, a theoretical computation of the electronic structure of C_{60} can be performed. There is good agreement between the group-lattice results and experiments. [5, 6]

The *twisted* group lattices form a particular class of general group lattices. To define the d -dimensional twisted group lattice, we must specify G_d and NN . The group G_d has d generators, denoted by x_1, x_2, \dots, x_d . They satisfy several relations. First, $x_i^{L_i} = e$, for $i = 1, 2, \dots, d$, where L_i are positive integers. Secondly, it is useful to define

$$z_{ij} \equiv x_j^{-1} x_i^{-1} x_j x_i, \quad \text{for } i < j. \quad (1.1)$$

The z_{ij} commute with the x_k , and consequently with themselves. In addition, $z_{ij}^{N_{ij}} = e$, where N_{ij} are non-negative integers. Consistency requires that N_{ij} and N_{ji} divide L_i for $j \neq i$. Let us summarize these statements:

Firstly, the group G_d associated with the d -dimensional twisted group lattice is

$$G_d = \left\{ x_1^{n_1} x_2^{n_2} \dots x_d^{n_d} \prod_{i < j} z_{ij}^{n_{ij}} \text{ , such that } n_i = 0, 1, \dots, L_i - 1, \right. \\ \left. n_{ij} = 0, 1, \dots, N_{ij} - 1, \ x_j x_i x_j^{-1} x_i^{-1} = z_{ij} \text{ for } i < j, \ x_k z_{ij} = z_{ij} x_k, \right. \\ \left. z_{ij} z_{kl} = z_{kl} z_{ij}, \ x_i^{L_i} = z_{ij}^{N_{ij}} = e, \ N_{ij} \text{ divides } L_i \text{ and } L_j \text{ for all } i < j \right\} \quad (1.2)$$

We remark that the group G_d is an iterated semidirect product of cyclic groups. First observe that G_d contains an abelian normal subgroup, $A_d \triangleleft G_d$, with

$$A_d = \left\{ x_d^{n_d} \prod_{i=1}^{d-1} z_{id}^{n_{id}} \right\} = \mathbf{Z}_{L_d} \otimes \mathbf{Z}_{N_{1d}} \otimes \dots \otimes \mathbf{Z}_{N_{d-1,d}} \quad (1.3)$$

From this it is clear that the quotient group G_d/A_d is just the group associated to the twisted group lattice in $d-1$ dimensions,

$$G_d/A_d = G_{d-1} \subset G_d \quad (1.4)$$

One says that G_d is an extension of G_{d-1} by A_d . Since G_{d-1} is also a subgroup of G_d , one has $G_{d-1} \cap A_d = \{e\}$, and G_d is the semi-direct product of G_{d-1} and A_d ,

$$G_d = G_{d-1} \cdot A_d \quad (1.5)$$

By iterating this construction, one arrives at

$$G_d = (\dots((G_1 \cdot A_2) \cdot A_3) \cdot \dots) \cdot A_d \quad (1.6)$$

with $G_1 = \mathbf{Z}_{L_1}$ and A_i given by Eq. (1.3).

Secondly, the nearest-neighbor set NN associated with G_d is chosen to be

$$NN = \{x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_d, x_d^{-1}\} \quad (1.7)$$

Apparently, a different group lattice is obtained for different values of N_{ij} and L_i . The center of the group G_d is generated by the z_{ij} . The number of elements in G is

$$o(G_d) = \left(\prod_{i=1}^d L_i \right) \left(\prod_{i < j} N_{ij} \right) \quad (1.8)$$

The general form of an element is

$$g = x_1^{n_1} x_2^{n_2} \dots x_d^{n_d} \prod_{ij} z_{ij}^{n_{ij}} \quad (1.9)$$

where n_i ranges from 0 to L_i-1 and n_{ij} ranges from 0 to $N_{ij}-1$. Associate with a site g of the form in Eq. (1.9) the point

$$(n_1, n_2, \dots, n_d; n_{12}, n_{13}, \dots, n_{d-1d}) \quad , \quad (1.10)$$

in the $d(d+1)/2$ dimensional hypercubic lattice. We call the first d coordinates the space coordinates and the last $d(d-1)/2$ coordinates the internal coordinates.

The nearest neighbors of g are obtained by multiplying by x_i or x_i^{-1} . When g in Eq. (1.9) is multiplied by x_i , the exponent of x_i increases by one and becomes n_i+1 .¹ This corresponds to a movement of one unit in the i th direction. When g in Eq. (1.9) is multiplied by x_i^{-1} , the exponent of x_i decreases by one and becomes n_i-1 . This corresponds to a movement of one unit in the negative i th direction. If all $N_{ij} = 1$, then the regular periodic hypercubic lattice in d dimensions is obtained, i. e. $A_i = \mathbf{Z}_{L_i}$ and $G_d = \bigotimes_{i=1}^d \mathbf{Z}_{L_i}$. The twisted group lattices, for which $N_{ij} \geq 2$, differ from regular lattices in that one does not necessarily return to the same element when going around a closed path. For example, going around an elementary plaquette in the i - j plane starting at the origin corresponds to $x_j^{-1}x_i^{-1}x_jx_i$, for $i < j$. This path does not return to the origin e but to z_{ij} . In general, a path starting at the origin returns to the origin if and only if the region projected onto each of the i - j planes has an area which is 0 (mod N_{ij}). When particles propagate on a twisted group lattice, this constraint needs to be taken into account.

The main goal of the current work is to solve the general d -dimensional group lattice. This requires that one obtains the irreducible representations of G_d in Eq. (1.2). In principle, all irreducible representations of G_d can be constructed from those of G_{d-1} by using the method of induced representations.^[7] However, the explicit iteration of this method based on Eq. (1.6) becomes rather involved. In this paper, we have developed a different and more direct approach. In Sect. 2, unitary representations are found. In Sect. 3, it is proven that the representations obtained in Sect. 2 are irreducible and complete. With this solution, one can use the formulas in Refs. [1, 8] to compute interesting results. The free propagation of particles on a d -dimensional twisted group lattice can be solved. If interacting field theories are put on these group lattices, a perturbative expansion can be performed.

The cases of $d = 2$, $d = 3$ and $d = 4$ have been solved in Refs. [3, 4, 8]. The solution method of the current work serves to reproduce the results for $d = 2$, $d = 3$ and $d = 4$ in a unified framework. For general d , we uncover an interesting connection with alternating bilinear forms on free \mathbf{Z}_N -modules of finite rank. We develop this and the corresponding mathematics in Sect. 2. Because of the interesting physical character of the twisted group lattices there are likely to be applications in physics.

¹The exponents of the z_{ij} may also change.

One possibility worth exploring is generalized spin. The gamma matrices Γ_i of the euclidean Dirac equation satisfy $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = \delta_{ij}$. This system of equations is closely related to the twisted group lattice with $N_{ij} = 2$ for all $i < j$. Indeed, gamma matrix structure has been shown to arise naturally for this case. [8] Generalized gamma matrices for which $\Gamma_i \Gamma_j \pm \Gamma_j \Gamma_i = \delta_{ij}$ for various signs may be of interest. The results of Sect. 2 allow one to construct the irreducible representations for this algebra. Equations such as $\Gamma_i \Gamma_j + \exp(i\phi_{ij}) \Gamma_j \Gamma_i = 0$, where $\exp(i\phi_{ij})$ are roots of unity, are of interest in parastatistics. Again, the results of Sect. 2 may be of use.

At an intermediate step in our solution method, we are lead to the equation

$$\Gamma_i \Gamma_j = \Gamma_j \Gamma_i \exp(2\pi i n_{ij}/N) \quad , \quad (1.11)$$

where N and $n_{ij} = -n_{ji}$ are integers, and where Γ_i are matrices. The equations in Eq. (1.11) enter in several mathematical and physical contexts. When Γ_i are regarded as elements of an abstract algebra, (1.11) corresponds to the quantum hyperplane, which arises in regard to quantum groups. [9, 10, 11, 12] Hence, matrix solutions to (1.11) are matrix representations of the quantum hyperplane algebra. When the Γ_i are $N \times N$ matrices, where N is the parameter on the right-hand side of Eq. (1.11), there are several applications: (i) the solution to self-dual Yang-Mills equations on a hypertorus with twisted boundary conditions [13], (ii) Witten's work [14] on constraints on supersymmetry breaking, (iii) the twisted Eguchi-Kawai model [15] and (iv) the twisted Eguchi-Kawai model at finite temperature [16, 17]. Eq. (1.11) is also relevant for certain d -dimensional crystals with screw dislocations. [4]

Problem (i) was considered by G. 't Hooft in connection with quark confinement in an $SU(N)$ gauge theory. In Ref. [18], solutions were found in $d = 4$ for the special case in which N is not divisible by a prime-number squared. Complete solutions in arbitrary dimensions were found in refs. [19, 20]. Our results for the intermediate problem governed by Eq. (1.11) are equivalent to those in refs. [19, 20].

The solution to Eq. (1.11) for Γ_i of arbitrary size makes use of $N \times N$ matrices $P_{(N)}$ and $Q_{(N)}$ satisfying

$$P_{(N)} Q_{(N)} = Q_{(N)} P_{(N)} \exp\left(\frac{2\pi i}{N}\right) \quad . \quad (1.12)$$

An explicit realization is [21]

$$P_{(N)} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad ,$$

$$Q_{(N)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & e^{2\pi i/N} & 0 & 0 & \dots & 0 \\ 0 & 0 & e^{4\pi i/N} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e^{2(N-1)\pi i/N} \end{pmatrix}, \quad (1.13)$$

where the subscript (N) indicates the size of the matrix. The $N \times N$ identity matrix is denoted by $I_{(N)}$.

Because of the need for elaborate notation, we summarize some additional conventions here. When curly brackets $\{ \}$ appear around a variable with ij indices, it means that ij range over the $d(d-1)/2$ values $ij = 12, 13, \dots, d-1d$. Integers k_i and k_{ij} are used for representation labels. The letter p indicates a prime number. The integer ℓ labels different prime sectors. These and other superscripts are enclosed in parenthesis to avoid possibly confusion with an exponent. The letters r, s, t and u , with or without subscripts and superscripts, are reserved for integer powers. The reader should be aware that the use of the exponents r and u differs from Ref. [8]. Representations are denoted by (r) and (s) and are enclosed in parenthesis to distinguish them from the exponents r and s . For the corresponding dimension and character, the letters d and χ are used. A capital greek Ω indicates a two-form. The associated antisymmetric matrix is denoted by ω . The standard one-form basis is dx^1, dx^2, \dots, dx^d . The greek letters α and β stand for general one-forms. The corresponding component, i. e., the coefficient of dx^i , is denoted by using the equivalent latin index a_i and b_i . With a subscript i or ij , n stands for a power of a group element factor as in Eq. (1.9). Without a subscript, it labels different tensor product sectors. In the ℓ th prime sector, n ranges from 1 to N_ℓ . One should not confuse N_ℓ with the twisting integers N_{ij} which determine the group G_d in Eq. (1.2). Modular arithmetic is indicated with the symbol “mod”. Because the construction procedure is algebraically tedious, we have provided an example of the method in an appendix.

II. Irreducible Representation Construction

This section finds the complete set of irreducible representations for the d -dimensional twisted group lattice defined by the set $\{N_{ij}, L_i\}$. The representations are determined by the two sets of integers k_{ij} , for $1 \leq i < j \leq d$, and k_i , for $i = 1, \dots, d$. Hence, $d(d+1)/2$ integers specify a representation (r) . The k_{ij} are “internal momenta” associated with the z_{ij} group elements and range from 0 to $N_{ij}-1$. The k_i are space-time momenta and take on values from 0 to L_i-1 . However, some sets of k_i lead to equivalent representations. Equivalent sets are related by a group E which is specified below. It turns out that the dimension $d_{(r)}$ of (r) depends only on the ratios k_{ij}/N_{ij} and not on the k_i .

Since the z_{ij} commute with all elements, the corresponding representation matrices can be chosen to be diagonal

$$z_{ij} \rightarrow I_{(d(r))} \exp \left(2\pi i \frac{k_{ij}}{N_{ij}} \right) . \quad (2.1)$$

The phases in Eq. (2.1) are fixed by the requirement that $z_{ij}^{N_{ij}}$ be the identity element. The matrix representations of x_j are of the form^[4]

$$x_j \rightarrow \Gamma_j \exp \left(2\pi i \frac{k_j}{L_j} \right) , \quad (2.2)$$

where the Γ_i are $d_{(r)} \times d_{(r)}$ matrices satisfying

$$\Gamma_j \Gamma_i = \Gamma_i \Gamma_j \exp \left(2\pi i \frac{k_{ij}}{N_{ij}} \right) , \quad \text{for } 1 \leq i < j \leq d , \quad (2.3)$$

and

$$\Gamma_i^{L_i} = 1 , \quad \text{for } i = 1, 2, \dots, d . \quad (2.4)$$

The key problem is to find Γ_i satisfying Eqs. (2.3) and (2.4). Note that the k_i no longer appear.

For the rest of this section, fix the values of the $\{k_{ij}\}$. Since it is only the ratio k_{ij}/N_{ij} which enters Eqs. (2.1) and (2.3), we reduce this fraction by removing common factors:

$$k_{ij}/N_{ij} = k'_{ij}/N'_{ij} , \quad (2.5)$$

where

$$\gcd(k'_{ij}, N'_{ij}) = 1 . \quad (2.6)$$

When $k_{ij} = 0$, we set $k'_{ij} = 0$ and define $N'_{ij} = 1$. Let N' be the least common multiple of the N'_{ij} ,

$$N' \equiv \text{lcm}(\{N'_{ij}\}) . \quad (2.7)$$

At this point we use the prime factorization idea of Ref. [8]. Factorize N' and the N'_{ij} into primes p_ℓ using

$$N' = \prod_{\ell=1}^L p_\ell^{s_\ell} , \quad \text{where } s_\ell \geq 1 , \quad (2.8)$$

and

$$N'_{ij} = \prod_{\ell=1}^L p_\ell^{t_{ij}^{(\ell)}} , \quad \text{where } t_{ij}^{(\ell)} \geq 0 . \quad (2.9)$$

The L prime numbers which enter in N' are p_1, p_2, \dots, p_L . The integer $t_{ij}^{(\ell)}$ is defined to be zero when p_ℓ does not appear in N'_{ij} . Eq. (2.7) implies that s_ℓ is the maximum value of the $t_{ij}^{(\ell)}$ as ij ranges over $1 \leq i < j \leq d$, i. e.,

$$s_\ell = \max_{\{ij\}} \{t_{ij}^{(\ell)}\} \quad . \quad (2.10)$$

The ratio in Eq. (2.5) can be written as a sum of “prime fractions” via

$$\frac{k'_{ij}}{N'_{ij}} = \sum_{\ell=1}^L \frac{\omega_{ij}^{(\ell)}}{p_\ell^{s_\ell}} \quad . \quad (2.11)$$

Eq. (2.11) defines the integers $\omega_{ij}^{(\ell)}$. It may happen that the fraction $\omega_{ij}^{(\ell)}/p_\ell^{s_\ell}$ can be reduced, in which case $p \mid \omega_{ij}^{(\ell)}$, but if $t_{ij}^{(\ell)} = s_\ell$ for a particular value of ij , then, as consequence of Eqs. (2.8)–(2.10), $\gcd(\omega_{ij}^{(\ell)}, p_\ell) = 1$. In short, there is at least one $\omega_{ij}^{(\ell)}$ that is relatively prime to p_ℓ .

The results in the previous paragraph allow one to factorize the Γ_i . Write Γ_i as a tensor product of L factors via

$$\Gamma_i = \Gamma_i^{(1)} \otimes \Gamma_i^{(2)} \otimes \dots \otimes \Gamma_i^{(L)} \quad . \quad (2.12)$$

If

$$\Gamma_j^{(\ell)} \Gamma_i^{(\ell)} = \Gamma_i^{(\ell)} \Gamma_j^{(\ell)} \exp \left(2\pi i \frac{\omega_{ij}^{(\ell)}}{p_\ell^{s_\ell}} \right) \quad (2.13)$$

and

$$\left(\Gamma_i^{(\ell)} \right)^{L_i} = 1 \quad , \quad (2.14)$$

then Eqs. (2.3) and (2.4) are satisfied as a consequence of Eqs. (2.11)–(2.14), as one can easily check. The converse is also true. In summary, the problem has been reduced to finding representations of Eqs. (2.13) and (2.14), i. e. of a single prime-factor sector ℓ .

Since it is sufficient to consider a single sector, let us temporarily simplify notation by doing away with the index ℓ . Define

$$\omega_{ij} \equiv \omega_{ij}^{(\ell)} \quad , \quad s \equiv s_\ell \quad , \quad p \equiv p_\ell \quad , \quad (2.15)$$

so that

$$\Gamma_j^{(\ell)} \Gamma_i^{(\ell)} = \Gamma_i^{(\ell)} \Gamma_j^{(\ell)} \exp \left(2\pi i \frac{\omega_{ij}}{p^s} \right) \quad , \quad 1 \leq i < j \leq d \quad . \quad (2.16)$$

The ω_{ij} are integers modulo p^s . One has

$$\gcd(\{\omega_{ij}\}, p) = 1 \quad \text{for at least one value of } ij \quad . \quad (2.17)$$

It is convenient to define

$$\omega_{ji} \equiv -\omega_{ij} \pmod{p}, \quad \text{for } i < j \quad . \quad (2.18)$$

Then, the integers ω_{ij} comprise an antisymmetric $d \times d$ matrix, ω . Such a matrix can be associated with a formal two-form Ω defined by

$$\Omega \equiv \sum_{1 \leq i < j \leq d} \omega_{ij} dx^i \wedge dx^j \quad . \quad (2.19)$$

From Eq. (2.17), one concludes $\Omega \not\equiv 0 \pmod{p}$.

We now begin to decompose the two-form Ω . More precisely, we shall show that there exist $2M$ independent one-forms $\alpha^{(1)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(2)}, \dots, \alpha^{(M)}, \beta^{(M)}$, such that

$$\Omega = \alpha^{(1)} \wedge \beta^{(1)} + \alpha^{(2)} \wedge \beta^{(2)} + \dots + \alpha^{(M)} \wedge \beta^{(M)} + \Omega' \pmod{p^s} \quad . \quad (2.20)$$

Here, M is the smallest integer such that

$$\Omega^{\wedge(M+1)} \equiv \underbrace{\Omega \wedge \Omega \wedge \dots \wedge \Omega}_{M+1 \text{ times}} = 0 \pmod{p} \quad , \quad (2.21)$$

and

$$\Omega' = 0 \pmod{p} \quad . \quad (2.22)$$

We note that $2M$ is equal to the rank of the matrix $\omega \pmod{p}$. It makes sense to consider the rank of a matrix modulo p because the integers modulo p constitute a field. It is also a well-known result that the rank of an antisymmetric matrix is an even integer. If we write

$$\Omega' \equiv \sum_{1 \leq i < j \leq d} \omega'_{ij} dx^i \wedge dx^j \quad , \quad (2.23)$$

then Eq. (2.22) implies that

$$p \mid \omega'_{ij}, \quad \text{for all } i < j \quad . \quad (2.24)$$

We express Eq. (2.24) compactly as $p \mid \Omega'$. When we say that one-forms $\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(N)}$ are independent, we mean that the only solution to the equation $\sum_{n=1}^N c_n \omega^{(n)} = 0 \pmod{p}$ is $c_n = 0 \pmod{p}$.²

For $s = 1$, Eq. (2.20) reduces to the standard decomposition of a two-form over a number field (in this case \mathbf{Z}_p). When $s > 1$, it is non-standard because the abelian ring \mathbf{Z}_{p^s} contains zero divisors, namely p^u for $u = 1, \dots, s-1$. By hiding the zero divisor part of Ω in Ω' , we can apply the standard decomposition to $\Omega - \Omega'$.

²Equivalently, $\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(N)}$ are independent if and only if $\omega^{(1)} \wedge \omega^{(2)} \wedge \dots \wedge \omega^{(N)} \not\equiv 0 \pmod{p}$.

A constructive proof of Eqs. (2.20)–(2.22) is as follows. If $M = 0$, let $\Omega' = \Omega$. Then, there is nothing to show: If $M = 0$ then $\Omega = 0 \pmod{p}$ so that $\Omega' = 0 \pmod{p}$. Thus, assume $M \geq 1$. Since $\text{rank } \omega \pmod{p} = 2M$, there exists a least one entry ω_{ij} such that $\omega_{ij} \not\equiv 0 \pmod{p}$. By relabelling indices we may assume that $ij = 12$ without loss of generality. Let

$$\begin{aligned} a_1^{(1)} &\equiv 1, & a_2^{(1)} &= b_1^{(1)} \equiv 0, \\ a_i^{(1)} &\equiv -\omega_{2i}/\omega_{12} \pmod{p^s}, & i &\geq 3, \\ b_i^{(1)} &\equiv \omega_{1i}, & i &\geq 2. \end{aligned} \quad (2.25)$$

In defining $a_i^{(1)}$, it makes sense to divide by ω_{12} , because ω_{12}^{-1} exists since $\gcd(\omega_{12}, p) = 1$. Take

$$\alpha^{(1)} \equiv \sum_{i=1}^d a_i^{(1)} dx^i, \quad \beta^{(1)} \equiv \sum_{i=1}^d b_i^{(1)} dx^i. \quad (2.26)$$

Note that $\gcd(b_2^{(1)}, p) = \gcd(\omega_{12}, p) = 1$. Define the remainder two-form $\hat{\Omega}$ by

$$\hat{\Omega} = \Omega - \alpha^{(1)} \wedge \beta^{(1)}. \quad (2.27)$$

A short computation shows that

$$\hat{\Omega} = \sum_{3 \leq i < j \leq d} \hat{\omega}_{ij} dx^i \wedge dx^j, \quad (2.28)$$

so that $\hat{\Omega}$ involves only the differentials dx^3, dx^4, \dots, dx^d . Repeat the previous construction for $\hat{\Omega}$ and continue until the process terminates. The process must terminate because the rank of ω is finite. Eventually, some remainder matrix has zero rank modulo p . One then obtains the desired result in Eq. (2.20). After a relabelling of the index i for dx^i , one sees that $\alpha^{(m)}$ involves the differentials $dx^{2m-1}, dx^{2m+1}, dx^{2m+2}, \dots, dx^d$, and that $\beta^{(m)}$ involves the differentials $dx^{2m}, dx^{2m+1}, dx^{2m+2}, \dots, dx^d$. Hence, the forms $\alpha^{(1)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(2)}, \dots, \alpha^{(M)}, \beta^{(M)}$ are independent modulo p . As a consequence, their total wedge product is non-zero modulo p , proving that, modulo p , $\Omega^{\wedge M} \neq 0$ but $\Omega^{\wedge(M+1)} = 0$. This completes the proof.

An examination of this proof reveals that it is both inductive and constructive. Take $a_i^{(m)}, b_i^{(m)}, m = 1, \dots, M$, to be

$$\alpha^{(m)} \equiv \sum_{i=1}^d a_i^{(m)} dx^i, \quad \beta^{(m)} \equiv \sum_{i=1}^d b_i^{(m)} dx^i. \quad (2.29)$$

Define d -component vectors $\vec{a}^{(m)}$ and $\vec{b}^{(m)}$ by

$$\vec{a}^{(m)} \equiv (a_1^{(m)}, a_2^{(m)}, \dots, a_d^{(m)}) , \quad \vec{b}^{(m)} \equiv (b_1^{(m)}, b_2^{(m)}, \dots, b_d^{(m)}) .$$

Then, the $2M$ vectors $\vec{a}^{(m)}, \vec{b}^{(m)}$ are independent modulo p . The rank of ω modulo p is the number of $\vec{a}^{(m)}$ and $\vec{b}^{(m)}$ vectors, as one can verify. Hence, this is indeed $2M$ modulo p . We call the process of obtaining $\alpha^{(1)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(2)}, \dots, \alpha^{(M)}, \beta^{(M)}$ the *Symplectic Construction* for Ω .

The decomposition of Ω , as in Eq. (2.20), is not yet complete since, modulo \mathbf{Z}_{p^s} , the remainder Ω' is nonzero. However, we may continue after dividing Ω' by p , which must be possible according to Eq. (2.22). Repeated applications of Eqs. (2.20)–(2.22) leads to the result that any two-form Ω (as in Eq. (2.19)) can be decomposed as

$$\Omega = \sum_{n=1}^N p^{u_n} \alpha^{(n)} \wedge \beta^{(n)} \pmod{p^s} \quad , \quad (2.30)$$

where $0 \leq u_n \leq s-1$, $u_1 = 0$, and $\alpha^{(n)}, \beta^{(n)}$ are independent one-forms.

It is noteworthy that the $d \times d$ matrix of row vectors $\vec{a}^{(1)}, \vec{b}^{(1)}, \dots, \vec{a}^{(N)}, \vec{b}^{(N)}$ (filled up with additional null vectors if $2N < d$) constitutes essentially the transformation matrix X employed in refs. [19, 20].

For a proof, apply the symplectic construction to Ω to obtain

$$\Omega \equiv \Omega_0 = \alpha^{(1)} \wedge \beta^{(1)} + \alpha^{(2)} \wedge \beta^{(2)} + \dots + \alpha^{(M_0)} \wedge \beta^{(M_0)} + \Omega'_0 \pmod{p^s} \quad . \quad (2.31)$$

Put

$$u_1 = u_2 = \dots = u_{M_0} = 0 \quad .$$

Note that $M_0 \geq 1$ because $\Omega_0 \neq 0 \pmod{p}$. Let

$$\Omega_1 = \Omega'_0 / p \pmod{p^{s-1}} \quad .$$

It makes sense to divide by p because $p \mid \Omega'_0$ since $\Omega'_0 = 0 \pmod{p}$, i. e. every element of the matrix ω'_0 is divisible by p . Apply the symplectic construction to Ω_1 :

$$\Omega_1 = \alpha^{(M_0+1)} \wedge \beta^{(M_0+1)} + \dots + \alpha^{(M_0+M_1)} \wedge \beta^{(M_0+M_1)} + \Omega'_1 \pmod{p^{s-1}} \quad .$$

It may be that $M_1 = 0$ because $\Omega_1 = 0 \pmod{p}$ already; then $\Omega_1 = \Omega'_1$. If $M_1 > 0$, put

$$u_{M_0+1} = u_{M_0+2} = \dots = u_{M_0+M_1} = 1 \quad .$$

Since $\Omega'_1 = 0 \pmod{p}$, one can define

$$\Omega_2 = \Omega'_1 / p \pmod{p^{s-2}}$$

and apply the symplectic construction to Ω_2 , and so on. At each stage, M_u counts the number of u_n -values equal to u . The process must terminate with $\Omega'_{s-1} = 0$ and $\sum_{u=0}^{s-1} M_u \equiv N$, because the maximum number of independent one-forms is d . Hence,

$N \leq \frac{d}{2}$ for d even and $N \leq \frac{d-1}{2}$ for d odd. By these means, symplectic constructions for Ω_u , $u = 0, \dots, s-1$, are obtained, with $2M_u = \text{rank } \omega_u \pmod{p}$. One reconstructs Ω'_u from Ω_{u+1} by using $\Omega'_u = p\Omega_{u+1} \pmod{p^{s-u}}$ for $u = s-1, \dots, 0$, beginning with $\Omega'_{s-1} = 0$. Then, one recursively expresses these two-forms as summands $\alpha^{(n)} \wedge \beta^{(n)}$ to arrive at Eq. (2.30). The proof is completed.

Let us now restore the index ℓ . For each ℓ there is a two-form $\Omega^{(\ell)}$ of the form

$$\Omega^{(\ell)} = \sum_{n=1}^{N_\ell} p_\ell^{u_{\ell n}} \alpha^{(\ell n)} \wedge \beta^{(\ell n)} \pmod{p_\ell^{s_\ell}} \quad . \quad (2.32)$$

We define $a_i^{(\ell n)}$ and $b_i^{(\ell n)}$ via

$$\alpha^{(\ell n)} \equiv \sum_{i=1}^d a_i^{(\ell n)} dx^i, \quad \beta^{(\ell n)} \equiv \sum_{i=1}^d b_i^{(\ell n)} dx^i \quad . \quad (2.33)$$

Actually, for a fixed ℓ , by relabelling the index i , one may assume, without loss of generality, that the sum in i for $\alpha^{(\ell n)}$ starts at $2n-1$ and the sum in i for $\beta^{(\ell n)}$ starts at $2n$ and that

$$a_{2n-1}^{(\ell n)} = 1, \quad a_{2n}^{(\ell n)} = 0, \quad \gcd(b_{2n}^{(\ell n)}, p) \neq 0 \quad . \quad (2.34)$$

These results follow from the details of the symplectic construction and imply that, within the ℓ th sector, the vectors $\vec{a}^{(\ell n)}, \vec{b}^{(\ell n)}$, for $n = 1, \dots, N_\ell$, are independent modulo p .

We now claim that

$$\Gamma_i^{(\ell)} = \bigotimes_{n=1}^{N_\ell} \left(Q_{(p_\ell^{r_{\ell n}})}^{a_i^{(\ell n)}} P_{(p_\ell^{r_{\ell n}})}^{b_i^{(\ell n)}} \exp \left[-i\pi (L_i - 1) \frac{a_i^{(\ell n)} b_i^{(\ell n)}}{p_\ell^{r_{\ell n}}} \right] \right) \quad (2.35)$$

is a solution to Eqs. (2.13) and (2.14). Here, $\bigotimes_{n=1}^{N_\ell}$ indicates a tensor product of N_ℓ matrices, and the powers $a_i^{(\ell n)}$ and $b_i^{(\ell n)}$ of the matrices $Q_{(p_\ell^{r_{\ell n}})}$ and $P_{(p_\ell^{r_{\ell n}})}$ are given by Eq. (2.33). The dimension d_ℓ of the representation for the ℓ th sector is

$$d_\ell = \prod_{n=1}^{N_\ell} p_\ell^{r_{\ell n}}, \quad (2.36)$$

where

$$r_{\ell n} \equiv s_\ell - u_{\ell n} \quad . \quad (2.37)$$

The phase factor in Eq. (2.35) ensures that Eq. (2.14) holds. It is a straightforward computation to verify Eq. (2.13). Hence, Eq. (2.35) is indeed a representation of the $\Gamma_i^{(\ell)}$ matrices of the ℓ th sector. When inserted in Eq. (2.12), a unitary representation

of the group G_d associated with the d -dimensional twisted group lattice is obtained via Eqs. (2.1) and (2.2). The total dimension $d_{(r)}$ of the representation is

$$d_{(r)} = \prod_{\ell=1}^L d_{\ell} \quad . \quad (2.38)$$

In the next section, we show that Eq. (2.35) gives a complete set of irreducible representations. Hence, our solution for $\Gamma_i^{(\ell)}$ is unique up to equivalence.

A different representation is obtained for different values of the k_{ij} . This is not the case for the k_i , however, since some sets $\{k_i\}$ lead to equivalent representations. Such $\{k_i\}$ are related by a group E . Hence, the (equivalence classes of) irreducible representations are uniquely labelled by $\{k_i\} \pmod{E}$ and $\{k_{ij}\}$.

We define a group E_{ℓ} , which is associated with the ℓ th sector of the tensor product in Eq. (2.12), by specifying its generators. There are $2N_{\ell}$ generators, and they act on the n th factor in Eq. (2.35) by conjugation. They are

$$\begin{aligned} E_P^{(\ell n)} : & \text{conjugation by } P_{(p_{\ell}^{r_{\ell n}})} \text{ on the } n\text{th factor in the } \ell\text{th sector} \quad , \\ E_{Q^{-1}}^{(\ell n)} : & \text{conjugation by } Q_{(p_{\ell}^{r_{\ell n}})}^{-1} \text{ on the } n\text{th factor in the } \ell\text{th sector} \quad , \end{aligned} \quad (2.39)$$

where $n = 1, 2, \dots, N_{\ell}$. A simple calculation shows that the effect of these conjugations is to shift the momenta k_i in the following way:

$$\begin{aligned} \text{for } E_P^{(\ell n)} : \quad k_i & \rightarrow k_i + \frac{a_i^{(\ell n)} L_i}{p_{\ell}^{r_{\ell n}}} \pmod{L_i} \quad , \\ \text{for } E_{Q^{-1}}^{(\ell n)} : \quad k_i & \rightarrow k_i + \frac{b_i^{(\ell n)} L_i}{p_{\ell}^{r_{\ell n}}} \pmod{L_i} \quad . \end{aligned} \quad (2.40)$$

Sets of momenta $\{k_i\}$ which are related by repeated shifts of Eq. (2.40) lead to equivalent representations. Because the $\vec{a}^{(\ell n)}$ and $\vec{b}^{(\ell n)}$ are independent, the number of identifications made under E_{ℓ} in Eq. (2.40) is $\prod_{n=1}^{N_{\ell}} p_{\ell}^{2r_{\ell n}} = d_{\ell}^2$.

The full group E is the tensor product of the groups E_{ℓ} . Since different primes are relatively prime, the identifications in Eq. (2.40) for different ℓ are independent. The total number of identifications under E is

$$\prod_{\ell=1}^L d_{\ell}^2 = d_{(r)}^2 \quad . \quad (2.41)$$

III. Proof of Irreducibility and Completeness

To prove irreducibility and completeness of the representations, it suffices^[22] to verify two equations. The first is

$$\sum_{(r)} d_{(r)}^2 = o(G_d) \quad , \quad (3.1)$$

and the second is the orthogonality of characters

$$\sum_{g \in G_d} \chi^{(r)}(g) \chi^{(s)}(g^{-1}) = \delta^{(r)(s)} o(G_d) \quad . \quad (3.2)$$

Recall that the representations are characterized by the $\{k_{ij}\}$, which range from 0 to $N_{ij}-1$, and the $\{k_i\}$ modulo E .

To prove Eq. (3.1), note that the number of identifications under E is $d_{(r)}^2$, according to Eq. (2.41). The dimension of a representation depends only on the $\{k_{ij}\}$ and not on the $\{k_i\}$. Since each k_i ranges from 0 to L_i-1 , there are

$$\frac{\prod_{i=1}^d L_i}{d_{(r)}^2}$$

independent values of the $\{k_i\}$. Consequently,

$$\sum_{(r)} d_{(r)}^2 = \sum_{\{k_{ij}\}} \sum_{\{k_i\} \pmod{E}} d_{(r)}^2 = \sum_{\{k_{ij}\}} \frac{\prod_{i=1}^d L_i}{d_{(r)}^2} d_{(r)}^2 = \left(\prod_{i < j} N_{ij} \right) \left(\prod_i L_i \right) = o(G_d) \quad . \quad (3.3)$$

This is Eq.(3.1).

It is more tedious to test Eq. (3.2). To distinguish the two representations in Eq. (3.2), we use the superscripts (r) and (s) . Hence, the representations (r) and (s) associated with the characters correspond to the sets $\{\{k_{ij}^{(r)}\}, \{k_i^{(r)}\}\}$ and $\{\{k_{ij}^{(s)}\}, \{k_i^{(s)}\}\}$. We also append superscripts (r) and (s) to the other quantities obtained in the constructions of Sect. 2.

Express the element g as in Eq. (1.9). The sum over g in Eq. (3.2) then becomes a sum over the powers n_{ij} and n_i :

$$\begin{aligned} & \sum_{g \in G_d} \chi^{(r)}(g) \chi^{(s)}(g^{-1}) \\ &= \sum_{\{n_i\}} \sum_{\{n_{ij}\}} \chi^{(r)}(x_1^{n_1} x_2^{n_2} \dots x_d^{n_d} \prod_{ij} z_{ij}^{n_{ij}}) \chi^{(s)}(x_d^{-n_d} \dots x_2^{-n_2} x_1^{-n_1} \prod_{i < j} z_{ij}^{-n_{ij}}) \quad . \end{aligned} \quad (3.4)$$

Equation (2.1) implies

$$\chi^{(r)}(g) \chi^{(s)}(g^{-1}) \propto \exp \left[2\pi i \sum_{i < j} \frac{\Delta k_{ij} n_{ij}}{N_{ij}} \right] \quad , \quad (3.5)$$

where $\Delta k_{ij} \equiv k_{ij}^{(r)} - k_{ij}^{(s)}$. By summing over the n_{ij} in Eq. (3.4), one concludes that

$$\sum_{\{n_{ij}\}} \chi^{(r)}(g) \chi^{(s)}(g^{-1}) = 0, \quad \text{unless } \Delta k_{ij} = 0 \text{ for all } i < j. \quad (3.6)$$

If $k_{ij}^{(r)} \neq k_{ij}^{(s)}$ for some ij , then the representations (r) and (s) are different, Eq. (3.4) is zero, and Eq. (3.2) is satisfied. Hence, for the remainder of this section we may assume that

$$k_{ij}^{(r)} = k_{ij}^{(s)}, \quad \text{for all } i < j. \quad (3.7)$$

With all $\Delta k_{ij} = 0$, the sum over n_{ij} in Eq. (3.4) produces a factor of

$$\sum_{\{n_{ij}\}} 1 = \prod_{i < j} N_{ij}. \quad (3.8)$$

Since the k_{ij} are now the same for both representations, almost all the quantities defined in Sect. 2 are the same for (r) and (s) :

$$\begin{aligned} k'_{ij}{}^{(r)} &= k'_{ij}{}^{(s)}, & N'_{ij}{}^{(r)} &= N'_{ij}{}^{(s)}, & \text{for } 1 \leq i < j \leq d, \\ \Gamma_j^{(\ell)(r)} &= \Gamma_j^{(\ell)(s)}, & \Omega^{(\ell)(r)} &= \Omega^{(\ell)(s)}, & \text{for each } \ell \text{ sector} \end{aligned} \quad (3.9)$$

As a consequence, the exponents in Eqs. (2.29) and (2.37) as well as the vector components in Eq. (2.33) in each ℓ sector are the same:

$$\begin{aligned} u_{\ell n}^{(r)} &= u_{\ell n}^{(s)}, & r_{\ell n}^{(r)} &= r_{\ell n}^{(s)}, \\ a_i^{(\ell n)(r)} &= a_i^{(\ell n)(s)}, & b_i^{(\ell n)(r)} &= b_i^{(\ell n)(s)}, & \text{for } i = 1, 2, \dots, d \end{aligned} \quad (3.10)$$

for all n . The only difference between the matrices of the two representations is in the k_j in the exponent of Eq. (2.2).

The matrices $P_{(N)}$ and $Q_{(N)}$ in Eq. (1.13) satisfy the property that $\text{Tr}(P_{(N)}^n Q_{(N)}^{n'})$ vanishes unless $n = 0 \pmod{N}$ and $n' = 0 \pmod{N}$. Likewise in a tensor product, $\text{Tr}(\otimes_m Q_{(N_m)}^{n_m} P_{(N_m)}^{n'_m})$ is zero unless $n_m = 0 \pmod{N_m}$ and $n'_m = 0 \pmod{N_m}$ for all m . Using Eqs. (2.2), (2.12) and (2.35), one concludes that the nonzero terms in the n_i sums in Eq. (3.4) occur exactly when

$$\sum_{i=1}^d a_i^{(\ell n)} n_i = \sum_{i=1}^d b_i^{(\ell n)} n_i = 0 \pmod{p_\ell^{r_{\ell n}}}, \quad (3.11)$$

for all n and all ℓ . These constraints need to be imposed on the n_i in the sums of Eq. (3.4). When they are satisfied, one has

$$\chi^{(r)}(g) \chi^{(s)}(g^{-1}) = d_{(r)}^2 \exp \left[2\pi i \sum_{i=1}^d \frac{\Delta k_i n_i}{L_i} \right], \quad (3.12)$$

where $\Delta k_i \equiv k_i^{(r)} - k_i^{(s)}$. Incorporating Eq. (3.8), we have

$$\sum_{g \in G_d} \chi^{(r)}(g) \chi^{(s)}(g^{-1}) = d_{(r)}^2 \delta_{k_{ij}^{(r)}, k_{ij}^{(s)}} \left(\prod_{i < j} N_{ij} \right) \sum'_{\{n_i\}} \exp \left[2\pi i \sum_{i=1}^d \frac{\Delta k_i n_i}{L_i} \right], \quad (3.13)$$

where the prime indicates that the n_i sums are constrained by Eq. (3.11).

In Ref. [8], a trick was found to handle the constraints in Eq. (3.11). By introducing additional summation variables $m^{(\ell n)}$ and $\overline{m}^{(\ell n)}$, the constraints can be implemented by inserting into free n_i sums the factor

$$\prod_{\ell} \left\{ \prod_{n=1}^{N_{\ell}} \left(\frac{1}{p_{\ell}^{r_{\ell n}}} \sum_{m^{(\ell n)}=1}^{p_{\ell}^{r_{\ell n}}} \exp \left[-2\pi i \frac{m^{(\ell n)}}{p_{\ell}^{r_{\ell n}}} \sum_{i=1}^d a_i^{(\ell n)} n_i \right] \right) \right\} \times \\ \prod_{\ell} \left\{ \prod_{n=1}^{N_{\ell}} \left(\frac{1}{p_{\ell}^{r_{\ell n}}} \sum_{\overline{m}^{(\ell n)}=1}^{p_{\ell}^{r_{\ell n}}} \exp \left[-2\pi i \frac{\overline{m}^{(\ell n)}}{p_{\ell}^{r_{\ell n}}} \sum_{i=1}^d b_i^{(\ell n)} n_i \right] \right) \right\}. \quad (3.14)$$

Hence, we insert Eq. (3.14) in Eq. (3.13) and sum over the n_i freely. When the n_i sums are done before the $m^{(\ell n)}$ and $\overline{m}^{(\ell n)}$ sums³, one finds a non-zero result if and only if

$$\Delta k_i = L_i \sum_{\ell} \left(\sum_{n=1}^{N_{\ell}} \left(\frac{m^{(\ell n)} a_i^{(\ell n)}}{p_{\ell}^{r_{\ell n}}} + \frac{\overline{m}^{(\ell n)} b_i^{(\ell n)}}{p_{\ell}^{r_{\ell n}}} \right) \right) \pmod{L_i}, \quad (3.15)$$

for $i = 1, 2, \dots, d$. Recall that different sectors ℓ are associated with different primes, which are, of course, relatively prime. Combining this fact with an explicit examination of $a_i^{(\ell n)}$ and $b_i^{(\ell n)}$ of Sect. 2 reveals that there is at most one $m^{(\ell n)}$ and one $\overline{m}^{(\ell n)}$ which solve Eq. (3.15). Hence, at best, one term among the auxiliary summation variables $m^{(\ell n)}$ and $\overline{m}^{(\ell n)}$ contributes. On the other hand, if the momenta $k_i^{(r)}$ and $k_i^{(s)}$ differ as in Eq. (3.15), then they are related by an element of E involving the generators raised to $m^{(\ell n)}$ th and $\overline{m}^{(\ell n)}$ th powers,

$$\prod_{\ell} \left\{ \prod_{n=1}^{N_{\ell}} \left(\left[E_P^{(\ell n)} \right]^{m^{(\ell n)}} \left[E_{Q^{-1}}^{(\ell n)} \right]^{\overline{m}^{(\ell n)}} \right) \right\}, \quad (3.16)$$

as can be seen from Eqs. (2.39) and (2.40). We have thus shown that $(r) \sim (s)$ if a non-zero result in Eq. (3.2) is to be obtained.

It remains to check whether the normalization in Eq. (3.2) is correct when (r) is equivalent to (s) . When $(r) \sim (s)$, there is a unique non-zero term in the $m^{(\ell n)}$ and

³The interchange of the order of sums is permitted because all sums involve a finite number of terms.

$\overline{m}^{(\ell n)}$ sums which contributes. It satisfies Eq. (3.15). For this term, the phase factors in Eqs. (3.13) and (3.14) cancel. Then the sums over the n_i merely produce

$$\prod_{i=1}^d L_i \quad ,$$

and the $1/p_\ell^{r_{\ell n}}$ factors in Eq. (3.14) give

$$\prod_{\ell} \frac{1}{\prod_{n=1}^{N_\ell} p_\ell^{2r_{\ell n}}} = \frac{1}{d_{(r)}^2} \quad .$$

When these results are inserted into Eq. (3.13), one finds

$$\begin{aligned} \sum_{g \in G_d} \chi^{(r)}(g) \chi^{(s)}(g^{-1}) &= d_{(r)}^2 \delta^{(r)(s)} \frac{\prod_{i=1}^d L_i}{d_{(r)}^2} \left(\prod_{i < j} N_{ij} \right) \\ &= \delta^{(r)(s)} \left(\prod_{i=1}^d L_i \right) \left(\prod_{i < j} N_{ij} \right) = \delta^{(r)(s)} o(G_d) \quad . \end{aligned} \quad (3.17)$$

This is Eq. (3.2).

IV. Conclusion

In this work, we have obtained the representations of the group associated with the general d -dimensional twisted group lattice. A proof that all the irreducible representations have been found was given in Sect. 3.

Our work solves the general d -dimensional twisted group lattice. The partition function Z for a free charged bosonic theory is given by Eqs. (4.5)–(4.7) of Ref. [8], i. e.,

$$Z = \prod_{(r)} \left[\det \left(\sum_{h \in NN_e} \lambda_h D^{(r)}(h) \right) \right]^{-d_{(r)}} \quad , \quad (4.1)$$

where

$$\begin{aligned} \sum_{h \in NN_e} \lambda_h D^{(r)}(h) &= \lambda_e I_{(d_{(r)})} \\ &+ \sum_{j=1}^d \left[\lambda_h \Gamma_j \exp\left(\frac{2\pi i k_j}{L_j}\right) + \lambda_{h^{-1}} \Gamma_j^\dagger \exp\left(\frac{-2\pi i k_j}{L_j}\right) \right] \quad , \end{aligned} \quad (4.2)$$

and where $NN_e = \{e\} \cup NN$, λ_e is a mass parameter, and the λ_h for $h \in NN$ are hopping parameters. Here, $\{k_{ij}, k_i\}$ are the representation labels associated with (r) .

The gamma matrices Γ_j in Eq. (4.2) are given in Eqs. (2.12) and (2.35). The product in Eq. (4.1) is over all the irreducible representations (r) of G_d , i. e.,

$$\prod_{(r)} = \prod_{\{k_{ij}=0\}}^{\{N_{ij}-1\}} \prod_{\{k_i\}} \pmod{E} \quad , \quad (4.3)$$

where the group E is specified in Eq. (2.40). The partition function Z corresponds to a gas of closed oriented loops, where the loops can be regarded as particle trajectories. The loops must satisfy the constraint that the area projected onto the i - j plane has zero area $\pmod{N_{ij}}$, for all $i < j$. To perform a perturbative expansion of an interacting theory, one needs the propagator. It involves the inverse of the matrix in Eq. (4.2). The precise formula is given in Eq. (4.9) of Ref. [1]. The partition function for the free fermionic case is given by Eq. (4.1) with the exponent $-d_{(r)}$ replaced by $d_{(r)}$.

The solution methods obtained in Refs. [3, 4, 8] for the $d = 2$, $d = 3$ and $d = 4$ cases appear to be quite different. However, it can be verified that the two-form approach of the current work reproduces the constructions for these cases. Our method provides a unifying framework for understanding these systems for different dimensions.

The twisted lattices can be used as discrete models of Euclidean space-time.^[8] When all L_i are equal and all N_{ij} are equal, the twisted lattices possess the same discrete rotational symmetries as regular lattices. This was shown in Ref. [8]. Compared to regular lattices, discrete translations no longer commute. Hence, a twisted group lattice corresponds to a non-commutative geometry of a discrete Euclidean space-time.

Finally, we would like to remark that because Eq. (1.11) appears in many contexts, it is likely that our construction method for the case in which the dimension of the matrices Γ_i is arbitrary will find future uses.

In summary, we have solved the general d -dimensional twisted group lattice and provided a few applications. Further investigations will hopefully uncover additional uses for our research.

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Appendix

In this appendix, we illustrate the method of constructing the Γ_i of Sect. 2 for a six-dimensional twisted lattice. Since the Γ_i depend only on k'_{ij} and N'_{ij} , it suffices to specify these primed quantities. We select

$$\begin{aligned}
k'_{13} &= k'_{14} = k'_{23} = k'_{24} = k'_{26} = k'_{56} = 0 \quad , \\
k'_{12} &= k'_{15} = k'_{16} = k'_{25} = k'_{34} = k'_{35} = k'_{45} = k'_{46} = 1 \quad , \\
k'_{36} &= 3 \quad , \\
N'_{13} &= N'_{14} = N'_{23} = N'_{24} = N'_{26} = N'_{56} = 1 \quad , \\
N'_{12} &= N'_{15} = N'_{16} = N'_{25} = N'_{45} = N'_{46} = 2 \quad , \\
N'_{34} &= N'_{35} = N'_{36} = 4 \quad .
\end{aligned} \tag{A.1}$$

The least common multiple N' of the N'_{ij} is 4, so that we are dealing with a single prime case. It is therefore not necessary to specify the prime sector, and we drop the label ℓ . The quantities in Eq. (2.8) are

$$N' = 2^2 \quad , \quad p = 2 \quad , \quad s = 2 \quad . \tag{A.2}$$

From Eqs. (A.1) and (A.2), one obtains the ω_{ij} . Then, the matrix $\omega = (\omega_{ij})$ is computed from Eq. (2.15). One finds

$$\omega = \begin{pmatrix} 0 & 2 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 3 & 0 & 2 & 2 \\ 2 & 2 & 3 & 2 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 \end{pmatrix} \pmod{4} \quad . \tag{A.3}$$

The next step is to achieve the decomposition of Eq. (2.30). The entry ω_{34} satisfies $\omega_{34} \not\equiv 0 \pmod{2}$. For the symplectic construction, it was assumed that $\omega_{12} \not\equiv 0 \pmod{2}$. Hence, one performs the label interchanges $1 \leftrightarrow 3$, $2 \leftrightarrow 4$, constructs the $a_i^{(1)}$ and the $b_i^{(1)}$ using the method of the Lemma, and then interchanges the labels back. The results are $a_3^{(1)} = 1$, $a_4^{(1)} = 0$, $a_1^{(1)} = -\omega_{41}/\omega_{34} = 0$, $a_2^{(1)} = -\omega_{42}/\omega_{34} = 0$, $a_5^{(1)} = -\omega_{45}/\omega_{34} = -2 = 2 \pmod{4}$, $a_6^{(1)} = -\omega_{46}/\omega_{34} = -2 = 2 \pmod{4}$, $b_3^{(1)} = 0$, $b_4^{(1)} = \omega_{34} = 1$, $b_1^{(1)} = \omega_{31} = 0$, $b_2^{(1)} = \omega_{32} = 0$, $b_5^{(1)} = \omega_{35} = 1$, $b_6^{(1)} = \omega_{36} = 3$. Hence, the one-forms in Eq. (2.26) are

$$\alpha^{(1)} = dx^3 + 2dx^5 + 2dx^6 \quad , \quad \beta^{(1)} = dx^4 + dx^5 + 3dx^6 \quad , \tag{A.4}$$

and the corresponding vectors $\vec{a}^{(1)}$ and $\vec{b}^{(1)}$ read

$$\vec{a}^{(1)} = (0, 0, 1, 0, 2, 2) , \quad \vec{b}^{(1)} = (0, 0, 0, 1, 1, 3) \quad . \quad (\text{A.5})$$

Following Eq. (2.27), $\hat{\Omega} = \Omega - \alpha^{(1)} \wedge \beta^{(1)}$. A straightforward computation gives the corresponding matrix $\hat{\omega}$:

$$\hat{\omega} = \begin{pmatrix} 0 & 2 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \pmod{4} \quad . \quad (\text{A.6})$$

The entries involving indices 3 and 4 are zero, as expected since we have performed the construction procedure using ω_{34} .

Note that $\hat{\omega}$ is divisible by 2. Hence, the remainder form $\hat{\Omega}$ equals Ω'_0 , and

$$\Omega = \alpha^{(1)} \wedge \beta^{(1)} + \Omega'_0 \pmod{4} \quad . \quad (\text{A.7})$$

Following Eq. (2.30), we construct Ω_1 by dividing by p : $\Omega_1 = \Omega'_0/2$. The corresponding matrix $\omega_1 \equiv \tilde{\omega}$ is

$$\tilde{\omega} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \pmod{2} \quad . \quad (\text{A.8})$$

The entry $\tilde{\omega}_{12}$ satisfies $\tilde{\omega}_{12} \neq 0 \pmod{2}$. We use this entry to construct $a_i^{(2)}$ and $b_i^{(2)}$. The method of Sect. 2 gives $a_1^{(2)} = 1$, $a_2^{(2)} = 0$, $a_3^{(2)} = -\tilde{\omega}_{23}/\tilde{\omega}_{12} = 0$, $a_4^{(2)} = -\tilde{\omega}_{24}/\tilde{\omega}_{12} = 0$, $a_5^{(2)} = -\tilde{\omega}_{25}/\tilde{\omega}_{12} = -1 = 1 \pmod{2}$, $a_6^{(2)} = -\tilde{\omega}_{26}/\tilde{\omega}_{12} = 0$, $b_1^{(2)} = 0$, $b_2^{(2)} = \tilde{\omega}_{12} = 1$, $b_3^{(2)} = \tilde{\omega}_{13} = 0$, $b_4^{(2)} = \tilde{\omega}_{14} = 0$, $b_5^{(2)} = \tilde{\omega}_{15} = 1$, $b_6^{(2)} = \tilde{\omega}_{16} = 1$. These results are summarized as

$$\alpha^{(2)} = dx^1 + dx^5 , \quad \beta^{(2)} = dx^2 + dx^5 + dx^6 , \quad (\text{A.9})$$

or

$$\vec{a}^{(2)} = (1, 0, 0, 0, 1, 0) , \quad \vec{b}^{(2)} = (0, 1, 0, 0, 1, 1) \quad . \quad (\text{A.10})$$

The next remainder form, $\hat{\Omega}_1$, is calculated using $\hat{\Omega}_1 = \Omega_1 - \alpha^{(2)} \wedge \beta^{(2)}$. One finds that the corresponding matrix $\hat{\omega}_1$ is

$$\hat{\omega}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \pmod{2} , \quad (\text{A.11})$$

so that

$$\hat{\Omega}_1 = \alpha^{(3)} \wedge \beta^{(3)} , \quad (\text{A.12})$$

where

$$\alpha^{(3)} = dx^5 , \quad \beta^{(3)} = dx^6 , \quad (\text{A.13})$$

and

$$\vec{a}^{(3)} = (0, 0, 0, 0, 1, 0) , \quad \vec{b}^{(3)} = (0, 0, 0, 0, 0, 1) . \quad (\text{A.14})$$

The computation terminates since all two-forms have been decomposed completely.

In summary,

$$\Omega = \alpha^{(1)} \wedge \beta^{(1)} + 2 \left(\alpha^{(2)} \wedge \beta^{(2)} + \alpha^{(3)} \wedge \beta^{(3)} \right) \pmod{4} , \quad (\text{A.15})$$

where the $\alpha^{(n)}$ and the $\beta^{(n)}$ are given in Eqs. (A.4), (A.9) and (A.13). From Eq. (A.15), one concludes that $M_0 = 1$ and $M_1 = 2$, implying $u_1 = 0$ and $u_2 = u_3 = 1$. Hence, the powers $r_n = 2 - u_n$ are $r_1 = 2$, $r_2 = 1$, and $r_3 = 1$, and the sizes of the matrices in the tensor products are $2^2 = 4$, $2^1 = 2$ and $2^1 = 2$. The dimension of the representation is 16. Equation (A.15) is the result of the decomposition (2.30) for the two-form Ω associated with ω in Eq. (A.3).

The matrices Γ_i are constructed using Eq. (2.35):

$$\begin{aligned} \Gamma_1 &= I_{(4)} \otimes Q_{(2)} \otimes I_{(2)} , & \Gamma_2 &= I_{(4)} \otimes P_{(2)} \otimes I_{(2)} , \\ \Gamma_3 &= Q_{(4)} \otimes I_{(2)} \otimes I_{(2)} , & \Gamma_4 &= P_{(4)} \otimes I_{(2)} \otimes I_{(2)} , \\ \Gamma_5 &= Q_{(4)}^2 P_{(4)} \otimes Q_{(2)} P_{(2)} \otimes Q_{(2)} , & \Gamma_6 &= Q_{(4)}^2 P_{(4)}^3 \otimes P_{(2)} \otimes P_{(2)} . \end{aligned} \quad (\text{A.16})$$

The powers of the Q and P matrices for the n th tensor factor are the components of the vectors $\vec{a}^{(n)}$ and $\vec{b}^{(n)}$, which were given in Eqs. (A.5), (A.10) and (A.14). The phases in Eq. (2.35) for Γ_5 and Γ_6 in Eq. (A.16) can be dropped for this example because 4 must divide L_5 and L_6 . In other words, $\Gamma_i^{L_i} = I_{(16)}$ holds for the Γ_i in

Eq.(A.16) with or without the phase factor in Eq. (2.35). It is a useful exercise to verify that the Γ_i do indeed satisfy

$$\Gamma_j \Gamma_i = \Gamma_i \Gamma_j \exp \left(2\pi i \frac{k'_{ij}}{N'_{ij}} \right) \quad \text{for } j > i \quad ,$$

with the k'_{ij} and N'_{ij} chosen in Eq. (A.1).

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